

Traversable wormholes: minimum violation of null energy condition revisited

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It was argued in literature that traversable wormholes can exist with arbitrarily small violation of null energy conditions. We show that if the amount of exotic material near the wormhole throat tends to zero, either this leads to a horn instead of a wormhole or the throat approaches the horizon in such a way that infinitely large stresses develop on the throat.

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I. INTRODUCTION AND BASIC EQUATIONS

One of the key features of wormholes consists in that the null energy condition (NEC) should be violated in the vicinity of a throat [1]. The corresponding material has the property unusual for laboratory physics and is called "exotic". Although the inevitability of violation of NEC is well known in physics of wormholes, the extent to which it occurs is still being debated in literature. It was stated in [2] that amount of exotic matter needed to support wormholes can be made as small as one likes (see also [4] - [7]). Let we have the sequence of traversable wormholes depending of some parameter ε such that in the limit $\varepsilon \rightarrow 0$ the amount of exotic material tends to zero. According to [2], one is inclined to think that a configuration with arbitrary small but non-zero ε represents an usual traversable wormhole and, thus, some kind of discontinuity happens that separates the state with $\varepsilon = 0$ from those with $\varepsilon \neq 0$. By itself, this does not necessarily mean something wrong since topological properties of the configuration with $\varepsilon = 0$ and $\varepsilon \neq 0$ are qualitatively different, so one may or may not expect discontinuity. Indeed, a state with small $\varepsilon \neq 0$ connects two

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asymptotically flat universes whereas the state with $\varepsilon = 0$ does not.

Nonetheless, the work [2] left one key question open. Let us call, for brevity, a wormhole standard if 1) there is no horizon, 2) its geometry remains regular (Kretschmann scalar is finite), 3) the areal radius r as a function of the proper distance l grows away from the throat in some finite vicinity. It is not quite clear from the results of [2] whether or not the limiting configuration remains the standard wormhole when the amount of exotic material tends to zero. The aim of the present article is to show that the answer to this question is negative. In this sense, if we restrict ourselves by standard wormholes, an amount of the exotic material *cannot* be made arbitrarily small. We also find what metric appears in the limit $\varepsilon = 0$ depending on which of conditions 1) - 3) is violated.

It is worth noting another approach in which the constraints on the wormhole geometries are derived from quantum inequalities which null-contracted stress-energy tensor should obey, when averaged over a timelike worldline [3]. Then, it turns out that in the concrete models considered as candidates for arbitrarily small violation of NEC, the typical throat radius cannot be macroscopic. Meanwhile, in our approach we do not constraint the properties of the source (so, classical sources are, in principle, also allowed provided they are compatible with NEC violation) and appeal directly to geometrical consequences which follow directly from the definitions of wormholes and Einstein equations.

We consider spherically-symmetric metrics

$$ds^2 = -e^{2\Phi} dt^2 + \frac{dr^2}{V} + r^2 d\omega^2, \quad d\omega^2 = \sin^2 \theta d\phi^2 + d\theta^2, \quad V = 1 - \frac{b}{r} \quad (1)$$

where b and Φ are the shape and redshift functions, respectively. We assume that the stress-energy tensor has the diagonal form

$$T_\mu^\nu = \text{diag}(-\rho, p_r, p_t, p_t). \quad (2)$$

Then, it follows from the Einstein equations that

$$\rho = \frac{b'}{8\pi r^2}; \quad p_r = \frac{1}{8\pi} \left[-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} \right]; \quad p_t = \frac{1}{8\pi} \left\{ \left(1 - \frac{b}{r}\right) \left[\Phi'' + \Phi' \left(\Phi' + \frac{1}{r} \right) \right] - \frac{b' - \frac{b}{r}}{2r} \left(\Phi' + \frac{1}{r} \right) \right\}. \quad (3)$$

The $r - r$ component of the conservation law $T_{\mu;\nu}^\nu = 0$ that can be obtained from Einstein equations (3) reads

$$p_r' + \Phi' \rho + \left(\Phi' + \frac{2}{r} \right) p_r - \frac{2p_t}{r} = 0. \quad (4)$$

It is convenient to introduce the quantity

$$\xi \equiv 8\pi r^2(p_r + \rho) \quad (5)$$

and use, along with the coordinate r , also the proper distance

$$l = \int \frac{dr}{\sqrt{V}}. \quad (6)$$

Then one obtains from (1), (3) that

$$V = 1 - \frac{r_0}{r} - \frac{2m(r)}{r}, \quad m(r) = 4\pi \int_{r_0}^r \rho \bar{r}^2 d\bar{r}, \quad (7)$$

$$\xi = -V'r + 2\Phi'rV, \quad (8)$$

$$\frac{d^2r}{dl^2} = \frac{V'(r)}{2}. \quad (9)$$

At the throat $r = r_0$ we must have, by definition, $\frac{dr}{dl} = 0$, so that $b(r_0) = r_0$, and

$$\frac{d^2r}{dl^2}(r_0) = -\frac{\xi(r_0)}{2r_0}, \quad (10)$$

$$p_t(r_0) = \frac{1 - b'(r_0)}{16\pi r_0} [\Phi'(r_0) + \frac{1}{r_0}]. \quad (11)$$

We are interested in traversable wormholes, so the horizon is supposed to be absent, $\Phi(r_0)$ is finite.

II. MEASURING AMOUNT OF EXOTIC MATTER

Now we must choose the method to measure the degree of "exoticism". In general, NEC is violated on the throat and/or in some vicinity of it. Therefore, one can, instead of NEC itself, consider averaged null condition (ANEC) obtained by integration of NEC [8]. This procedure was somewhat changed in [2]. To gain information about the total amount of matter which violates NEC, it was suggested in [2] to consider the volume integral $I \equiv \int_{r_0}^{\infty} dr \xi$. In this connection, we must make a technical remark. The boundary term at infinity was lost in eq. (12) of [2] which is equal to $b(\infty)$ and, in general, does not vanish contrary to what is stated in [2]. What is more important, the definition of I should be, in our view, modified. In the form it was introduced in [2], the integral extends to infinity. Therefore, it may happen that $I > 0$ but there is a region of strong violation of NEC compensated by the contribution of

the normal matter. Meanwhile, it looks more natural to be interested in the contribution from the exotic region alone. Therefore, we will consider somewhat different definition

$$I \equiv \int_{r_0}^a dr \xi = 8\pi \int_{r_0}^a dr r^2 (p_r + \rho), \quad (12)$$

where it is assumed that the exotic matter fills the inner region $r_0 \leq r < a$, while the outer region $r \geq a$ is occupied by the normal matter with NEC satisfied or by vacuum ("normal region" for brevity). In doing so, we use the same integral measure as in [2] but restrict the integration by the exotic region. The appearance of the factor $dr r^2$ in the integral (12) is motivated by the analogy with the ADM mass formula as is explained in [2]. For our system, the quantity $\xi < 0$ for $r_0 \leq r < a$ and $\xi \geq 0$ for $r \geq a$. Actually, in concrete examples considered in [2] the value of I does not depend on which of two definitions is used since the outer region lies in vacuum and does not contribute to the integral I .

As $\xi \leq 0$ in the integrand, we have $I \leq 0$. Our goal is to elucidate whether and under which conditions one can obtain $I \rightarrow 0$. As we want to preserve regularity we consider everywhere finite ξ . Then $I \rightarrow 0$ entails that either 1) $a \rightarrow r_0$ (limits of integration shrink) or 2) $\xi \rightarrow 0$ (the integrand vanishes) in the whole exotic region (or both 1 and 2 hold). In other words, we can try to minimize either the size of the region or "exoticism" or violation of NEC in the relationship between p_r and ρ (i.e. in the equation of state). Let us discuss different cases separately.

III. MINIMIZING SIZE OF REGION WITH EXOTIC MATERIAL

A. Continuous distribution

1. $\xi(r_0) < 0$

Let us now assume that pressures p_r , p_t and the energy density ρ are continuous functions of r . Therefore, on the border between the exotic and normal regions,

$$\xi(a) = 0. \quad (13)$$

On the throat, it is supposed that NEC are violated as usual [1], [8], so $\xi(r_0) < 0$. We are interested in regular configurations. The function $\Phi(r)$ and its first and second derivatives are supposed to be bounded, unless the opposite is stated explicitly. In the limit

under discussion $\lim_{a \rightarrow r_0} \frac{d\xi}{dr}(a) = \lim_{a \rightarrow r_0} \frac{\xi(a) - \xi(r_0)}{a - r_0} \rightarrow \infty$. It follows also from (4) that p'_r is finite. Then, the only way to reconcile the finiteness of p'_r with divergency of ξ' consists in considering infinite ρ' . Let us consider, without a big loss of generality, the density profile of the form

$$\rho = \rho_0 + \Delta\rho \frac{(r - r_0)^n}{(a - r_0)^n} \quad (14)$$

for $r_0 \leq r \leq a$ and $\rho(r)$ is some smooth function with bound derivatives for $r > a$. It follows from (14) that $\rho(r_0) = \rho_0$, $\rho(a) = \rho_0 + \Delta\rho$. From (13), we have that $\rho(a) = -p_r(a)$. In the limit $a \rightarrow r_0$ $\rho'(a) \rightarrow \infty$, so we have a jump in ρ . (For $a - r_0$ small but non-zero the distribution of ρ is still continuous.) Due to continuity of p_r and the finiteness of p'_r , in this limit we obtain also that $\rho(a) = -p_r(r_0) + O(\delta)$, $\delta = a - r_0$. On the throat, eq. (3) entails a well-known equality $p_r(r_0) = -(8\pi r_0^2)^{-1}$ [1], [8]. Therefore, for small δ

$$\rho(a) = \frac{1}{8\pi a^2} + O(\delta). \quad (15)$$

How does the geometry look like in this limit? To answer this question, consider first the geometry for $r > a$. We assume that for $r \rightarrow a$ the corresponding function admits the Taylor expansion $V = V(a) + V'(a)(r - a) + \frac{V''(a)}{2}(r - a)^2 + O((r - a)^3)$ with finite coefficients. Then, it follows from (7) and (15) that $V(a) \rightarrow 0$ and $V'(a) \rightarrow 0$ in the limit $\delta \rightarrow 0$. As a result, the metric in the vicinity of $r = a$ looks like

$$ds^2 = -dt^2 \exp[2\Phi(a)] + \frac{2dr^2}{V''(a)(r - a)^2} + a^2 d\omega^2 \quad (16)$$

and represent a horn in the sense that the proper distance between $r = a$ and $r > a$ diverges. In doing so, the quantity $\exp[2\Phi(a)]$ does not vanish since there is no horizon by our assumption. The part of the manifold $r \geq a$ becomes geodesically complete and does not resemble a wormhole.

It is also instructive to trace what happens to the region $r_0 \leq r < a$ in this limit. Again, we assume the Taylor expandability in the vicinity of r_0 . Then, it is easy to obtain that for small δ the expansion reads

$$V = \frac{r - r_0}{r} Z, \quad (17)$$

$$Z = -\xi(r_0) - 8\pi r_0^2 \Delta\rho \frac{(r - r_0)^n}{(n + 1)\delta^n} + O(\delta^2). \quad (18)$$

Making a substitution $r = r_0 + \delta y$, where $0 \leq y \leq 1$ we obtain that

$$ds^2 = -dt^2 \exp[2\Phi(a)] + \delta \frac{dy^2}{f(y)} + r_0^2 d\omega^2, \quad (19)$$

$f(y) = -\xi(r_0)y - 8\pi r_0^2 \Delta \rho \frac{y^n}{(n+1)}$ does not contain δ . Then, it becomes obvious that the proper distance between the throat and the boundary $r = a$ between the normal and exotic regions is finite and, moreover, tends to zero, so that this part of space shrinks to a disc and, thus, is removed from the manifold.

2. Case $\xi(r_0) = 0$

Up to now, we assumed that $\xi(r_0) < 0$. Correspondingly, the asymptotic expansion of the metric coefficient began from the linear terms according to (18). The situation changes, if $\xi(r_0) = 0$. The exotic region $\xi(r) < 0$ is assumed to occupy the interval $r_0 < r < a$ and $\xi(a) = 0$. Let us try again to find the configuration with the almost zero violation of NEC. As now $\xi(r_0) = 0$, it follows from (8) that $V'(r_0) = 0$. It follows from (7) that this condition means $\rho = \rho_0 = (8\pi r_0^2)^{-1}$. We assume that in the vicinity of the throat we have the asymptotics

$$V = A(r - r_0)^{n+1} \quad (20)$$

with $A = \text{const}$ and $n > 0$. Instead of an usual minimum of the function $r(l)$ typical of $n = 0$, now we deal with the minimum of the higher order. We want to find a regular wormhole configuration so that we require that the proper distance to the throat be finite. Thus, $0 < n < 1$. This is similar to the case discussed in section 4.1 of [7].

Then, we obtain from (8) that

$$\xi = -r_0(n+1)A(r - r_0)^n + B(r - r_0)^{n+1} + \dots \quad (21)$$

The exact value of the coefficient B (as well as the coefficients at the higher degrees in the expansion) is irrelevant for us. What is important is the fact that A does not contain the small parameter, so that ξ has the general form $\xi = -A_1 y + A_2 y^m$, $m = \frac{n+1}{n} > 1$, $y = r - r_0$, A_1 and A_2 are finite, $A_1 > 0$. Then, it becomes clear that the function ξ attains its minimum at some r_1 with finite $r_1 - r_0$. In turn, this means that $a - r_0 > a - r_1$ is also finite and cannot be made arbitrarily small. Thus, although now the proper distance is finite, our efforts to achieve the almost zero violation of NEC failed again.

From the other hand, we may impose by brute force the condition $a \rightarrow r_0$, allowing sequence of configurations with more and more small A . As the point $r = a$ is supposed to be a regular point, we may exploit the Taylor expansion $V = V(a) + V'(a)(r - a) + V''(a)\frac{(r - a)^2}{2} +$

.... It follows from the condition $\xi(a) = 0$ and eq. (8) that $V'(a) \sim V(a) \rightarrow V(r_0) = 0$. Thus, only the term proportional to $(r - a)^2$ may survive here, so the proper distance to $r = a$ diverges.

The analysis becomes especially simple if $\Phi = 0$. Then, we have from (8) that

$$\xi = -V'r. \quad (22)$$

Then, V' changes the sign at $r = a$, so that $V' < 0$ everywhere in a normal region. It means that $V(r)$ decreases to zero that does not correspond to a wormhole configuration (by assumption, $V = 0$ at the throat r_0 only but $V > 0$ for $r > r_0$).

In the work [6] the trial form of the shape function was chosen so that $V = \exp[\frac{K}{(r-r_0)^n}]$ with $n \geq 1$. Then, it is obvious that V diverges so strongly near the throat that the proper distance $l \rightarrow \infty$ and we have a horn instead of a wormhole.

B. Jump of ξ

The above arguments do not work directly if we allow the jump of the quantity $\xi(r)$. Let this quantity change from $\xi(a-0) < 0$ in the exotic region to $\xi(a+0) > 0$ in the normal one. Then, for a close to r_0 it follows from (8) that V' changes by jump from positive to negative values at $r = a$, while the term proportional to V is negligible. However, if the function $V(r)$ is small and negative near $a \rightarrow r_0$ and has a negative finite derivative $V' = -\frac{\xi}{r} + O(V)$ for $r > a$ it means that the function V changes the sign at some $b = a + O(V)$ in obvious contradiction with the properties of wormhole metrics. If, instead, ξ changes by jump from $\xi(a-0) < 0$ to $\xi(a+0) = 0$, the derivative $V'(a) \rightarrow 0$ and we have a horn.

IV. MINIMIZING "EXOTICISM"

Let $\xi(r) \rightarrow 0$ inside some region including $r = r_0$. If $\xi = 0$ (or, equivalently, $p_r = -\rho$) it follows from the Einstein equations (3) that only two possibilities exist.

A. Horizons instead of traversable wormholes

First case: $\exp(2\Phi) = 1 - \frac{b}{r}$. In this case at the supposed throat $r = r_0$ where $b(r_0) = r_0$ the g_{00} component of the metric tensor in (1) vanishes as well in contradiction with the

assumption about the absence of the horizon. Therefore, this case should be rejected.

B. Horn-like configurations

Second case: $b = r$. Then, formally, $g_{11} = \infty$. Actually, this only means that the coordinate r is degenerate and cannot be used. If, instead, we use l , one can see that $\frac{dr}{dl} = 0$, so $r = r_0 = \text{const}$ inside the corresponding region. Thus, instead of the wormhole we have a horn. In principle, one can take a finite piece of this horn and glue it to the metric with $\frac{dr}{dl} > 0$ on the right side and $\frac{dr}{dl} < 0$ on the left side. In doing so, one can obtain so-called "null wormholes" (N-wormholes) [12]. Although inside the region where $p_r + \rho = 0$ NEC is satisfied on the verge, gluing to the normal matter on the borders needs the presence of exotic material and we return to the situation under discussion with all corresponding difficulties.

One can also consider the situation when $\xi(r) \rightarrow 0$ when $r \rightarrow r_0$ but does not vanish identically in some region. Then, r is also not identically constant. In the vicinity of r_0 we still have a horn (semi-infinite throat) with small derivatives of r . Indeed, it follows directly from (8), that in this limit $\frac{d^2 r}{dl^2}(r_0) \rightarrow 0$. In a similar way, one can show, assuming analyticity of $b(r)$, that all higher derivatives $\frac{d^n r}{dl^n} \rightarrow 0$. Actually, it means that r_0 is at infinite proper distance from any $r > r_0$. As a result, the spacetime with r varying from r_0 to the right infinity is geodesically complete. Therefore, our configuration cannot be considered as a wormhole.

As an example of such a configuration, we can point to the exact solutions in dilaton gravity [11]:

$$ds^2 = -dt^2 + \left(1 - \frac{r_0}{r}\right)^{-2} dr^2 + r^2 d\omega^2. \quad (23)$$

In this example $\xi = -2\frac{r_0}{r}(1 - \frac{r_0}{r}) \rightarrow 0$ when $r \rightarrow r_0$, $r - r_0 \simeq \exp(-\frac{l}{r_0})$ in this limit. Meanwhile, r is not identically constant and, moreover, $r \rightarrow \infty$ at asymptotically flat infinity. The spacetime with $r_0 \leq r < \infty$ is geodesically complete and no wormhole arises.

One can also consider the case when $\xi(r_0) \neq 0$ but is small. When, for non-zero $\xi < 0$ a wormhole configuration will be indeed possible. However, in the limit $\xi(r_0) \rightarrow 0$ it will be approaching the horn-like one closer and closer, with $\frac{d^2 r}{dl^2}(r_0)$ becoming smaller and smaller, so in the limit $\xi(r_0) = 0$ we return to the situation described above: either it will be a horn everywhere ($r = \text{const}$) or asymptotically for $r \rightarrow r_0$. Anyway, the limit of sequence of such

configuration does not represent a wormhole.

C. Combined case

One can also try to combine both factors and consider decreasing $\xi(r_0)$ along with decreasing $a - r_0$. Actually, the model of this kind was considered in [5] where it is claimed that the violation of NEC can be arbitrarily small but it was not discussed what happens to the wormhole metric in this limit. In our notations, this model can be described as follows. Let $\Phi = 0$, $b = 1 - k - \epsilon(r)$ for $r_0 \leq r \leq a$ where $\epsilon'(r_0) = 0$, $k \rightarrow 1$, $a \rightarrow r_0$ and $\epsilon(r)$ has the order $k - 1$ in this limit. Then, one can obtain from [5] or directly from (3) that $\rho + p_r = -\frac{1-k}{8\pi r_0^2} < 0$ inside the interval $[r_0, a]$ where it can be made arbitrarily small by taking sufficiently small $1 - k$. However, in this limit the proper distance between any $r > a$ and $r = a$ behaves like $l(a, r) = \frac{r-a}{\sqrt{1-k}}$ and diverges in this limit. As a result, we again obtain a geodesically complete spacetime with a horn instead of a wormhole. It was already mentioned in [3] that in this model the proper distance between the throat and the border at a behaves similarly like $l(r_0, a) = \frac{r_0-a}{\sqrt{1-k}}$, so that only fine-tuning between parameters may warrant the finite $l(r_0, a)$ in the limit under discussion. We would like to point out that even such a fine-tuning does not save the matter since $l(a, r)$ diverge even if $l(r_0, a)$ remains finite. Shortcomings of this model as well as of some other models also flaring outward too slowly (see [3] for their detailed criticism) were repaired in [9] where minimizing the size of exotic region was discussed without requirement of making this region arbitrarily small. Instead, some balance between this size and restrictions due to quantum inequalities [3],[10] was suggested. This issue, however, is beyond the scope of the present paper.

V. LIMITING CONFIGURATIONS WITH A HORIZON

Our goal is to try to find wormhole configurations with the minimum violation of NEC. We saw in previous sections that, typically, it is the flare out condition which is violated in the limit under consideration, so a wormhole becomes more and more extended and approaches the horn. It is seen from (8) that the first term is negative near the throat while the second one is positive and small, provided $\Phi' > 0$ is finite. Roughly speaking, what we did in previous section is the attempt to diminish the first negative term by diminishing V' .

This attempt resulted in the appearance of horns instead of throats since small V' entail large proper distances to the border between regions. Let us try another method to achieve positive ξ in the small vicinity of the throat: instead of diminishing the first negative term we can try to increase the second positive one. We assume that $V \sim r - r_0$ near the throat. If we still take Φ bounded near the throat the factor V will make the second term in (8) negligible. Instead, we should take $\Phi \sim \ln(r - r_0)$. As we want to have the throat, we need Φ to be finite on the throat. The natural choice is

$$\exp(\Phi) = \varepsilon + \lambda f(r), \quad r_0 \leq r \leq a, \quad \lambda > 0 \quad (24)$$

which is glued smoothly to the patch with $r > a$. If $\varepsilon = 0$ from the very beginning, we obtain the horizon. Then, the requirement of regularity along with the asymptotics $V \sim r - r_0$ leads to the condition $f \sim A f_0$, $f_0 = \sqrt{1 - \frac{r_0}{r}}$ (by rescaling λ we always may achieve $A = 1$). For simplicity and without big loss of generality, we restrict ourselves by the case when $f = f_0$, $V = 1 - \frac{r_0}{r}$ exactly and $\exp(\Phi) = \sqrt{V}$ for $r > a$. This is just the example "Specialization 2" from [2]. The properties of this model were also discussed in [3] where it was shown that quantum inequalities [3] constrain it in such a way that make it unrealistic. However, this does not exclude, in principle, exploiting some unusual classical source for this geometry. In our context, we are interested in inner properties of the system irrespective of how it is created and do not appeal to any numerical estimates.

In [2], the authors mainly discussed the behavior of bulk density and pressure, meanwhile now we will see that the crucial role is played by surface stresses. They appear on the boundary between two different regions "+" (right) and "-" (left) [13] with components S_0^0 and $S_2^2 = S_3^3$ where

$$8\pi S_0^0 = \frac{2}{r} \left[\left(\frac{dr}{dl} \right)_+ - \left(\frac{dr}{dl} \right)_- \right], \quad (25)$$

$$8\pi S_2^2 = \frac{1}{r} \left[\left(\frac{dr}{dl} \right)_+ - \left(\frac{dr}{dl} \right)_- \right] + \left(\frac{d\Phi}{dl} \right)_+ - \left(\frac{d\Phi}{dl} \right)_-. \quad (26)$$

As V is supposed to be continuous, $S_0^0 = 0$ everywhere. On the border $r = a$

$$8\pi S_2^2(a) = \frac{r_0 \varepsilon}{2\varepsilon_s^2 a^2}, \quad \varepsilon_s \equiv \sqrt{1 - \frac{r_0}{a}} \quad (27)$$

where we follow the notations of [2]. We emphasize that for the metric under discussion, there are also the surface stresses on the throat itself where two branches of $r(l)$ with opposite

signs of $\frac{dr}{dt}$ meet. Simple calculations give us

$$8\pi S_2^2(r_0) = \frac{\lambda}{r_0 \varepsilon} \quad (28)$$

Now we will show that the stresses (27) and (28) cannot be made finite simultaneously in the limit $a \rightarrow r_0$. Indeed, in this limit the quantity $\varepsilon_s \rightarrow 0$. If ε is fixed, $S_2^2(a) \rightarrow \infty$ that hardly can be accepted physically. One may try to repair this shortcoming and, instead, take simultaneously the limits $\varepsilon_s \rightarrow 0$, $\varepsilon \rightarrow 0$ in such a way that $\frac{\varepsilon}{\varepsilon_s^2}$ remains finite or even zero. However, in the limit $\varepsilon \rightarrow 0$, $S_2^2(r_0) \rightarrow \infty$ independently of the relationship between ε and ε_s .

VI. SUMMARY

It turned out that the statement of the kind "there are wormholes with arbitrarily weak violation of NEC" should be taken with great care. We have shown that the construction of traversable wormholes with arbitrarily small amount of the exotic material is achieved by expense of violation of some properties inherent to standard traversable wormholes. We considered several ways to achieve the arbitrarily small amount of exotic material for wormholes geometry which, actually, can be divided to two types. The first one consists in diminishing V' (then the density approaches the critical value $\rho_0 = (8\pi r_0^2)^{-1}$). Then, typically, instead of a wormhole we obtain a horn in this limit. It does not mean that such configurations have no physical sense. By contrary, in some situations it is the tube-like configurations which are placed to the forefront (for example, in the context of the problem of avoidance singularities [12], in investigating properties of phantom matter [14], etc.). However, in any case, they represent a special class of wormhole configurations which differ from the standard ones. The second type represents wormholes on the threshold of formation of a horizon. Then, infinitely large surfaces stresses appear on the throat and/or the boundary between exotic and normal regions, so the limit is singular and, in this sense, unphysical.

As the quantitative measure of "exoticism" we exploit the value of I according to the definition (12) used in [2]. If, instead of I we would consider the similar integral J over the proper distance l we would obtain that $[J] > [I]$ since $V < 1$. Therefore, as $[I]$ cannot be made arbitrarily small without violation of conditions 1) - 3) indicated in Introduction, the

same applies to J . In this sense, using J as the quantifier of exoticism only strengthens the conclusion that degree of exoticism cannot be made arbitrarily small for standard wormholes.

One can think that if an advanced civilization wants to construct a standard wormhole, it should not make lame excuses about "small amount" of exotic material and must do inevitable job of collecting and arranging this material in sufficient supply.

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